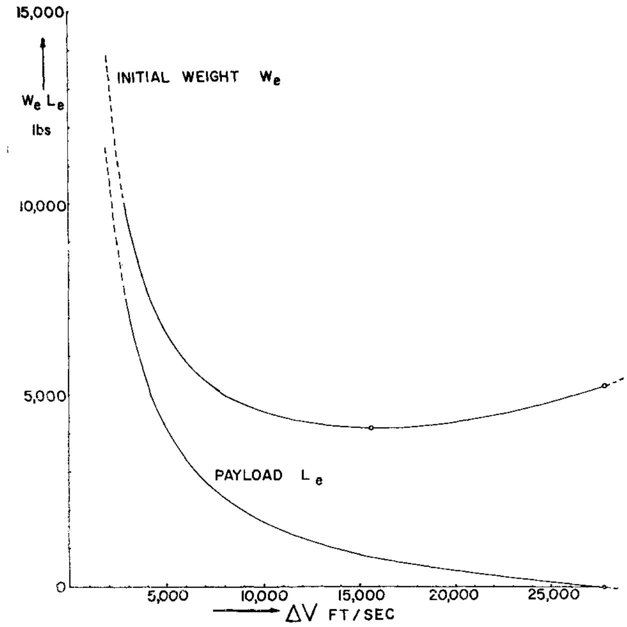

Fig. 1 Equivalence of nuclear and chemical drives.

rather than a turbopump for feeding propellants to the engine. Under this assumption, the following values were estimated for the constants: $A_n = 1100$ lb, $A_e = 150$ lb, $B_n = 0.20$, and $B_c = 0.10$. This gives $\alpha = 863.636$ lb, and $\beta = 0.90909$.

The curves separate the area of superiority of the chemical drive from the area of potential superiority of the nuclear drive in terms of ΔV and W or ΔV and L.

The curve for W_e has a minimum W_1 for $z = \frac{1}{2}\beta$:

$$W_1 = 4\alpha/\beta^2 = 4180 \text{ lb}$$

The corresponding velocity increase is $\Delta V_1 = 15614$ fps, so for rockets having an initial weight up to W_1, the chemical drive will always be better. For a heavier rocket, the velocity increase has to be considered. For an increase smaller than ΔV_1, W_e becomes larger and becomes infinite for $\Delta V = 0$. In other words, for a very small velocity increase the chemical drive is always better. This is obvious because then the weights of either nuclear or chemical propellants needed will be very small and there is nothing to offset the difference in engine weights. For values of ΔV larger than ΔV_1, W_e increases again, because of increasing propellant weight.

The curves break off for $\Delta V_2 = 27{,}635$ fps, at which the equivalent payload becomes zero. That does not mean that a single-stage rocket cannot carry any payload for a velocity increase larger than ΔV_2, but then the initial weight has to be larger than W_e; hence the nuclear rocket is better.

A payload of 200 lb of instruments will be large enough for most scientific measurements. With this value of L_e one finds a velocity increase of 23,676 fps. This is sufficient to send a vehicle from a 300-mile parking orbit around Earth to the vicinity of Saturn or well within the orbit of Mercury. This shows that for most unmanned missions in the solar system the chemical drive will be better than the nuclear-thermal and probably also the nuclear-electric drive that has a still higher engine weight.

The "equivalent" curves are, of course, dependent on the choice of the constants I_n, I_c, A_n, A_c, B_n, and B_c which affects the numerical values of W_1, ΔV_1, and ΔV_2. However, the minimum equivalent weight W_1 is found to vary only slightly within the practical limits of these constants.

Reference

[1] Penner, S. S., *Advanced Propulsion Techniques* (Pergamon Press, New York, 1961), p. 54.

Buckling of a Slender Prismatic Rod at High Impact Velocity

CHARLES W. BERT*
Battelle Memorial Institute, Columbus, Ohio

THIS paper is addressed to the following question: For given high-velocity impact conditions, can column buckling occur in a slender prismatic rod penetrating a target? Many analyses have been carried out for impact buckling of elastic rods.[1-4] Most of these have dealt with column action in rapid compression tests, rather than with the simpler question of whether or not buckling can initiate. The latter topic has been treated,[4] but the assumption of an elastic material limits the impact velocity v_0 to low values, because it is equal to $\epsilon_e c_e$, where ϵ_e is the elastic-limit strain, and c_e is the acoustic velocity of the rod. For most materials, $v_0 < 200$ fps for elastic conditions. Some plastic-range impact-buckling studies have been made,[5] but, apparently, none have been for high-velocity impact conditions, when the material obeys a nonlinear equation of state.

Here an analysis for an arbitrary material-behavior regime is carried out and applied to the high-velocity regime, using available data on rod penetration and mechanical equations of state.

Analysis

A slender rod with a uniform, compact cross section is struck axially at one end with a relative velocity v_0 at time $t = 0$. The impact force is not removed immediately but remains constant during a given time interval. A compressive pulse propagates along the rod at some velocity c (elastic, plastic, or shock, as the case may be) and has traveled a distance ct into the rod at time t. Assuming that penetration takes place at constant deceleration from the initial relative velocity v_0 to a final velocity $(v_0 - \Delta v)$ where Δv is the total velocity change for the given rod-target combination, the relative velocity during penetration is given by

$$v_p = v_0 - (t\Delta v/t_p) \tag{1}$$

where t_p is the time required for complete penetration.

The combination of rod penetration and disintegration of the rod end is assumed to be a steady-state process, and thus the instantaneous velocity of rod-end disintegration is $v_d = Av_p$, where A is a constant. Thus, the physical length subjected to axial compression at time t is given by

$$L_p = (c - Av_0)t + (A\Delta v t^2/2t_p) \tag{2}$$

The generalized Euler equation is appropriate for buckling of a slender rod[6]:

$$\sigma_{cr} = E_t(\pi k_{cr}/L)^2 \tag{3}$$

where σ_{cr} is the critical axial load per unit area at which buckling can occur; E_t is the tangent modulus (i.e., the slope of the stress-strain curve at stress σ_{cr}), k_{cr} is the critical minimum radius of gyration of the rod cross section (square root of the ratio of minimum centroidal moment of inertia to the cross-sectional area); and L is the equivalent buckling length

Received May 18, 1964; revision received June 12, 1964. These results were obtained in research conducted at Battelle for Advanced Research Projects Agency, Washington, D. C., under Contract SD-80. Computational aid by John Kasuba, University of Illinois graduate student and Battelle summer staff member, is gratefully acknowledged.

* Program Director, Solid and Structural Mechanics, Battelle Memorial Institute; now Associate Professor, School of Aerospace and Mechanical Engineering, University of Oklahoma, Norman, Okla. Member AIAA.

Table 1 High estimates of end-fixity coefficient (B) and maximum equivalent lengths (L) for four cases

Case	End clamped	t	L_p	B	$L = BL_p$
1	Opposite end	$\leq t_p$	$<L_a$	0.707	$0.707\ t_p [c - A(v_0 - \frac{1}{2}\Delta v)]$
2	Both ends	$\geq t_p$	$<L_a$	0.500	$0.5\ t_p [c - A(v_0 - \frac{1}{2}\Delta v)]$
3	Neither	$\leq t_p$	L_a	1.000	L_a
4	Impacted end	$\geq t_p$	L_a	0.707	$0.707\ L_a$

$L = BL_p$, where the constant B depends upon the boundary conditions at each end of the loaded portion.

The four distinct cases to be considered are shown in Table 1, where L_a is the actual length of the rod, and for B, conservative estimates given by the values appropriate for elastic buckling[6] are used.

From Table 1, it is seen that case 1 is more critical than case 2 and that case 3 is more critical than case 4; thus, only cases 1 and 3 need be considered. Case 1 is more critical than case 3 if

$$0.707\ t_p[c - A(v_0 - \tfrac{1}{2}\Delta v)] > L_a \tag{4}$$

Since t_p usually is not known, it is desirable to express it approximately in terms of thickness h penetrated as follows:

$$t_p = h/(v_0 - \tfrac{1}{2}\Delta v) \qquad v_0 > \tfrac{1}{2}\Delta v \tag{5}$$

(In order for penetration to occur, $v_0 > \frac{1}{2}\Delta v$.) Combining relations (4) and (5) gives

$$L_a < 0.707\ h[c - A(v_0 - \tfrac{1}{2}\Delta v)]/(v_0 - \tfrac{1}{2}\Delta v) \tag{6}$$

For the case of overkill, $\Delta v \ll v_0$; hence,

$$L_a < 0.707\ h[(c/v_0) - A] \tag{7}$$

Application to Solid Rods Impacting Thick Targets

Present data on rod penetration into thin-sheet targets are insufficient to predict penetration. However, in an unpublished analysis of unclassified data on penetration of solid round rods into thick ("semi-infinite") targets, H. S. Morton of Johns Hopkins University, Applied Physics Laboratory has shown that the distance h_{si} penetrated is given by the following equation:

$$h_{si} = B_n d(\rho/\rho_t)(L_a/d)^{2/3}\ (v_0/c_{te})^n \tag{8}$$

where d is the rod diameter, ρ_t is the target density, c_{te} is the elastic velocity in the target, $n = 1$ for $v_0 > v_T$, $n = 2$ for $v_0 < v_T$, v_T is a transitional velocity, and B_n is a constant depending upon n and the combination of rod and target materials. Also for a solid round rod, $k_{cr} = 0.3535\ d$.

The uniaxial equation of state can be written as[7]

$$\sigma = (\tfrac{2}{3})S_y + \sigma_h \tag{9}$$

where S_y is the yield strength and σ_h is the hydrostatic stress. Thus, the tangent modulus is

$$E_t \equiv d\sigma/d\epsilon = d\sigma_h/d\epsilon \tag{10}$$

The McQueen hydrostatic equation of state is commonly used

$$\sigma_h = K\epsilon/(1 - \lambda\epsilon)^2 \tag{11}$$

where K and λ are constants. By differentiating this σ_h with respect to ϵ and using Eq. (10), it follows that

$$K/E_t = [1 - (4\lambda\sigma/K)]^{1/2} \tag{12}$$

Table 2 Material properties

Material	Ref.	K, psi	λ
Steel	9	16.4×10^6	1.58
Aluminum	8	11.0×10^6	1.39

Table 3 Velocity v_0 above which buckling cannot take place

Rod material	Target material	v_0, fps
Steel	Steel	2800
Steel	Aluminum	4400
Aluminum	Steel	6100

Analyses of shock waves generated by one-dimensional impact of a rod into a thick target (of the same or another material), using the Rankine-Hugoniot relations and empirical equations of state, following Maiden et al.[8] gave σ as a function of v_0. The results show that the shock-wave propagation velocity c_s can be computed approximately by

$$c_s = c_{s0} + \beta v_0 \tag{13}$$

where c_{s0} and β are constants, depending upon the combination of materials.

In Eq. (8), it is necessary to use h_{si} instead of h and to let $v = v_0 - 0 = v_0$. Then $t_p = 2h_{si}/v_0$, and for case 1,

$$L = 1.414\ h_{si}[(c_{s0}/v_0) + \beta] \tag{14}$$

where h_{si} is given by Eq. (8). But $d = 2.828\ k_{cr}$. Then

$$K = (2.828/\pi)L(\sigma/K)^{1/2}[1 - (4\lambda\sigma/K)]^{1/4} \tag{15}$$

Thus,

$$d/L_a = \{(4/\pi)B_n(\rho/\rho_t)(v_0/c_{te})^n\ (c_{s0}/v_0) + \beta(\sigma/K)^{1/2}\ [1 - (4\lambda\sigma/K)]^{1/4}\}^{3/2} \qquad \text{(case 1)} \tag{16}$$

$$d/L_a = (2.828/\pi)(\sigma/K)^{1/2}[1 - (4\lambda\sigma/K)]^{1/4} \qquad \text{(case 3)} \tag{17}$$

Equations (16) and (17) both give positive values for d/L_a when $\sigma < K/4\lambda$.

Numerical Results

Table 2 gives the values of the material constants used, together with the sources. For a given strain, steel has a higher stress and the stress-strain curve has a greater slope than that for aluminum. Thus, when steel strikes steel, for a given impact velocity v_0, the stress is higher, and the curve of stress vs v_0 has a steeper slope than for steel impacting aluminum.

Table 3 gives results for three material combinations. Although these velocities are fairly high, they are well below the hypervelocity range (approximately 12,000 fps). Thus, they tend to make plausible occasional reports of straws penetrating trees during hurricanes.

Conclusions

In this approximate analysis, it is predicted that buckling of slender prismatic rods striking a semi-infinite medium cannot occur when the impact velocity exceeds a limiting velocity that depends only upon the materials involved. These velocities range from 2800 to 6100 fps for some metallic materials. In view of the uncertainties involved, it would be desirable to check this experimentally for at least one combination of materials.

References

[1] Koning, C. and Taub, J., "Impact buckling of thin bars in the elastic range hinged at both ends," NACA TM 748 (June 1934).

[2] Hoff, N. J., "The dynamics of the buckling of elastic columns," J. Appl. Mech. **18**, 68–74 (1951).

[3] Sevin, E., "On the elastic bending of columns due to dynamic axial forces including effects of axial inertia," J. Appl. Mech. **27**, 125–131 (1960).

[4] Gerard, G. and Becker, H., "Column behavior under conditions of impact," J. Aeronaut. Sci. **19**, 58–60 (1952).

[5] Hegglin, B., "Dynamic buckling of columns," Stanford Univ., SUDAER Rept. 129 (June 1962).
[6] Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability* (McGraw-Hill Book Co., Inc., New York, 1961), 2nd ed.
[7] Lundergan, C. D. and Herrmann, W., "Equation of state of 6061-T6 aluminum at low pressures," J. Appl. Phys. **34**, 2046–2052 (1963).
[8] Maiden, C. J., Gehring, J. W., and McMillan, A. R., "Investigation of fundamental mechanism of damage to thin targets by hypervelocity projectiles," General Motors Defense Research Labs., TR 63-208 (March 1963).
[9] Altshuler, L. V., Krupnikov, K. K., Ledener, B. N., Zhuchiklin, V. I., and Braznik, M. I., "Dynamic compressibility and equation of state of iron," Soviet Phys.-JETP **34**, 874—885 (1958).

Rapid Technique for Calculating Times of Maximum Thermal Stress in Simple Shapes

H. DERSHIN* AND R. G. SHOOK†
General Dynamics/Pomona, Pomona, Calif.

Nomenclature

a = inside radius
b = outside radius
c_p = specific heat
E = modulus of elasticity
$\dot{Q}$ = heat flux
$\dot{q}$ = total heat absorption
r = radius
T = temperature
t = time
V = volume
α = coefficient of thermal expansion
μ = indicates surface condition
ν = Poisson's ratio
ρ = density
σ_θ = hoop stress

A METHOD has been developed which permits rapid calculation of thermal stress in uniformly heated cylinders, spheres, or plates. In addition, the technique provides a means for determining the time at which maximum thermal stresses occur. Thus, a good deal of laborious calculation can be avoided.

Circumferential thermal stress in a hollow circular cylinder, heated uniformly, can be computed[1] from

$$\sigma_\theta = \left(\frac{E\alpha}{1-\nu}\right)\frac{1}{r^2}\left\{\frac{r^2+a^2}{b^2-a^2}\int_a^b rTdr + \int_a^r rTdr - r^2T\right\} \tag{1}$$

However, in most practical problems, the maximum stresses occur at one of the surfaces, where

$$\sigma_{\theta_a} = \frac{E\alpha}{1-\nu}\left\{\frac{2}{b^2-a^2}\int_a^b rTdr - T_a\right\}$$

$$\sigma_{\theta_b} = \frac{E\alpha}{1-\nu}\left\{\frac{2}{b^2-a^2}\int_a^b rTdr - T_b\right\} \tag{2}$$

But, the net rate of heat absorption in any volume is

$$\dot{q} = \int_V \rho c_p \frac{\partial T}{\partial t}\, dV \tag{3}$$

Received July 2, 1964.
* Staff Scientist. Member AIAA.
† Senior Research Engineer.

Or for a unit surface area, heated uniformly at the exterior by a flux Q_b and insulated at the interior

$$\dot{Q}_b = \int_a^b \rho c_p \frac{\partial T}{\partial t}\left(\frac{r}{b}\right) dr \tag{4}$$

Integrating over time, and substituting into Eq. (2),

$$\sigma_{\theta_a} = \sigma_{\theta_a}(0) + \frac{E\alpha}{1-\nu}\left\{\left(\frac{2b}{b^2-a^2}\right)\frac{1}{\rho c_p}\times \int_0^t \dot{Q}_b dt + T_a(0) - T_a(t)\right\}$$

$$\sigma_{\theta_b} = \sigma_{\theta_b}(0) + \frac{E\alpha}{1-\nu}\left\{\left(\frac{2b}{b^2-a^2}\right)\frac{1}{\rho c_p}\times \int_0^t \dot{Q}_b dt + T_b(0) - T_b(t)\right\} \tag{5}$$

where $\sigma_\theta(0)$ is the initial, steady-state stress (i.e., at $t = 0$). The same approach can be taken for a sphere and a plate restrained in bending. The general result is

$$\frac{\sigma_\mu(t) - \sigma_\mu(0)}{E\alpha/(1-\nu)} = \left\{\frac{nb^{n-1}}{b^n - a^n}\frac{1}{\rho c_p}\times \int_0^t \dot{Q}dt + T_\mu(0) - T_\mu(t)\right\} \tag{6}$$

where $n = 3, 2,$ or 1 for a sphere, cylinder, or plate.

Thus it is seen that the thermal stress at the surfaces of a uniformly heated sphere, cylinder, or plate can be determined from the thermal history by one simple integration. Also, the time of maximum thermal stress can be simply obtained by maximizing Eq. (6) with respect to time. Then, in the general case, the time of maximum stress is determined from the thermal history simply by finding the time at which

$$\dot{Q}_b = \left(\frac{b^n - a^n}{nb^{n-1}}\right)\rho c_p \frac{dT_\mu}{dt} \tag{7}$$

It should be emphasized that the foregoing equations are valid only for uniformly heated plates, cylinders, and spheres. This is because of the fact that Eqs. (6) and (7) contain, implicitly, solutions to the thermal conduction problems in only the bodies specified. It has been found, however, that this does not detract from their usefulness in estimating times of peak thermal stress in related configurations.

Reference

[1] Timoshenko, S. and Goodier, J. N., *Theory of Elasticity* (McGraw-Hill Book Co., New York, 1951), 2nd ed., pp. 408–409.

Technical Comment

Erratum: "Correlations for Theoretical Rocket Thrust with Shifting Expansion"

LEONARD GREINER*
Texaco Experiment Incorporated, Richmond, Va.

[J. Spacecraft Rockets **1**, 339–340 (1964)]

IN the 18th line of the second column of the above engineering note, the words "The foregoing" should read "Above."

Received June 23, 1964.
* Research Associate. Associate Fellow Member AIAA.